Bridge Cohomology

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Preliminaries

Complexes

For any $\Bbbk$-algebra A and any A-bimodule M we have the Hochschild and Bar cochain complexes $C^\bullet(A, M)$ and $C^\bullet_{\mathrm{bar}}(A, M)$, given by

$$C^\bullet(A, M) := M \xrightarrow{b} \mathrm{Hom}(A, M) \xrightarrow{b} \dots \xrightarrow{b} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{b} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{b} \dots$$

$$b\varphi(a_1, \dots, a_{n+1}) = a_1\varphi(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1}\varphi(a_1, \dots, a_n)a_{n+1}$$

Preliminaries

Complexes

For any $\Bbbk$-algebra A and any A-bimodule M we have the Hochschild and Bar cochain complexes $C^\bullet(A, M)$ and $C^\bullet_{\mathrm{bar}}(A, M)$, given by

$$C^\bullet_{\mathrm{bar}}(A, M) := M \xrightarrow{b'} \mathrm{Hom}(A, M) \xrightarrow{b'} \dots \xrightarrow{b'} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{b'} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{b'}$$

$$b'\varphi(a_1, \dots, a_{n+1}) = a_1\varphi(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1})$$

Acting on the modules $C^n_{(\mathrm{bar})}(A)$ we have the cyclic operator

$$\lambda : C^n_{(\mathrm{bar})}(A) \to C^n_{(\mathrm{bar})}(A)$$

$$\lambda\varphi(a_0, ..., a_n) = (-1)^n \varphi(a_n, a_0, ..., a_{n-1})$$

From this we can also create the norm operator

$$Q : C^n_{(\mathrm{bar})}(A) \to C^n_{(\mathrm{bar})}(A)$$

$$Q = \sum_{i=0}^{n} \lambda^i$$

Preliminaries

Maps

These operators have the following relationships with the differentials b and b':

$$(1-\lambda)b = b'(1-\lambda) \qquad Qb' = bQ$$

Thus we get chain maps

$$(1-\lambda) : C^\bullet(A) \to C^\bullet_{\mathrm{bar}}(A)$$

$$Q : C^\bullet_{\mathrm{bar}}(A) \to C^\bullet(A)$$

When $\Bbbk$ contains $\mathbb{Q}$ the sequence is exact

$$\ldots \xrightarrow{Q} C(A) \xrightarrow{1-\lambda} C_{\mathrm{bar}}(A) \xrightarrow{Q} C(A) \xrightarrow{1-\lambda} C_{\mathrm{bar}}(A) \xrightarrow{Q} \ldots$$

When A is a unital algebra, the Hochschild cohomology of A valued in M is defined as the cohomology of the complex $C^\bullet(A, M)$

$$HH^\bullet(A, M) := H^\bullet(C(A, M))$$

$$HH^\bullet(A) := H^\bullet(C(A, A^*))$$

For any algebra A (not necessarily unital) the bar cohomology of A (with coefficients in M) is the cohomology of the complex $C^\bullet_{\mathrm{bar}}(A)$ (or $C^\bullet_{\mathrm{bar}}(A, M)$)

$$HB^\bullet(A, M) := H^\bullet(C_{\mathrm{bar}}(A, M))$$

$$HB^\bullet(A) := H^\bullet(C_{\mathrm{bar}}(A, A^*))$$

When A is non-unital, we extend the functor $HH^\bullet$ by defining

$$HH^\bullet(A) = \ker(HH^\bullet(A_+) \to HH^\bullet(\mathbb{k}))$$

where A_+ is the algebra obtained by adjoining a unit to A.

Definition

For a unital algebra A, the *normalized Hochschild complex* is given by

$$\overline{C}^n(A) := \{\varphi \in C^n(A) \,|\, \varphi(a_0, \ldots, a_n) = 0 \text{ if some } a_i \in \Bbbk,\ 1 \leq i \leq n\}$$

$$0 \to \overline{C}^\bullet(A) \to C^\bullet(A) \to D^\bullet(A) \to 0$$

where the degenerate complex $D^n(A)$ is the set of functionals

$$D^n(A) := \{\varphi \,|\, \varphi(a_0, \ldots, a_n)\ a_i = 1,\ \text{for some } 1 \leq i \leq n\}$$

that extend to $C^n(A)$.

Define $\mathring{\mathbb{j}}_A$ as the following cokernel

$$0 \longrightarrow \overline{A}^* \longrightarrow A^* \longrightarrow \mathring{\mathbb{j}}_A \longrightarrow 0$$

determined by the image of the evaluation map $ev_1 : A^* \to \mathbb{k}$, where $\varphi \mapsto \varphi(1)$.

Definition

The *Reduced Hochschild cochain complex* for a unital algebra A is defined as the kernel

$$0 \to C^\bullet(A)_{\mathrm{red}} \to \overline{C}^\bullet(A) \to \mathring{\mathbb{j}}_A[0] \to 0.$$

where $\mathring{\mathbb{j}}_A[0]$ is the cocomplex with $\mathring{\mathbb{j}}_A$ in degree 0.

The *reduced Hochschild cohomology* is then

$$\overline{HH}^\bullet(A) := H^\bullet(C^\bullet(A)_{\mathrm{red}})$$

Corollary

For a non-unital algebra A, $HH^n(A) = H^n(C(A_+)_{\mathrm{red}})$.

Definition

The Connes complex $C_\lambda(A)$ is given as the kernel of $1 - \lambda$:

$$0 \to C_\lambda(A) \to C(A) \xrightarrow{1-\lambda} C_{\text{bar}}(A)$$

The "Cyclic" Cohomology of A is then $H^\bullet_\lambda(A) := H^\bullet(C_\lambda(A))$.

Cyclic Cohomology

Cyclic Bicomplex

Definition

For any algebra A the *cyclic cobicomplex* $CC^{\bullet\bullet}(A)$ is the bicomplex

$$
\begin{CD}
@A{b}AA @A{-b'}AA @A{b}AA @A{-b'}AA \\
C^2(A) @>{1-\lambda}>> C^2_{\mathrm{bar}}(A) @>{Q}>> C^2(A) @>{1-\lambda}>> C^2_{\mathrm{bar}}(A) @>{Q}>> \\
@A{b}AA @A{-b'}AA @A{b}AA @A{-b'}AA \\
C^1(A) @>{1-\lambda}>> C^1_{\mathrm{bar}}(A) @>{Q}>> C^1(A) @>{1-\lambda}>> C^1_{\mathrm{bar}}(A) @>{Q}>> \\
@A{b}AA @A{-b'}AA @A{b}AA @A{-b'}AA \\
C^0(A) @>{1-\lambda}>> C^0_{\mathrm{bar}}(A) @>{Q}>> C^0(A) @>{1-\lambda}>> C^0_{\mathrm{bar}}(A) @>{Q}>>
\end{CD}
$$

The n^{th} *cyclic cohomology of* A is then

$$HC^n(A) := H^n(\mathrm{Tot}\, CC^{\bullet\bullet}(A))$$

Proposition

Let $C_{\bullet\bullet} \to C'_{\bullet\bullet}$ be a map of bicomplexes which is a quasi-isomorphism when restricted to each column. Then the induced map on the total complexes is a quasi-isomorphism.

Thus, when $\mathbb{k}$ contains $\mathbb{Q}$ we have $H^\bullet_\lambda(A) \cong HC^\bullet(A)$.

The long exact sequence relating cyclic cohomology with Hochschild cohomology is given by the Gysin-Connes sequence:

$$\ldots \xrightarrow{I} HH^{n-1}(A) \xrightarrow{B} HC^{n-2}(A) \xrightarrow{S} HC^{n}(A) \xrightarrow{I} HH^{n}(A) \rightarrow \ldots$$

Where I is induced by the inclusion, $B = Qs(1 - \lambda)$, and S is induced by the cup product with the generator $\sigma \in H^{\bullet}(\Bbbk)$.

Theorems and Examples

Proposition (Connes)

Let τ be an $n+1$ linear functional on A. Then the following are equivalent:

1. *There is an n-dimensional cycle $(\Omega, d, \int)$ and a homomorphism $\rho : A \to \Omega^0$ such that*

$$\tau(a_0, ..., a_n) = \int \rho(a_0) d\rho(a_1)...d\rho(a_n)$$

2. *There exists a closed graded trace $\hat{\tau}$ of dimension n on $\Omega^*(A)$ such that*

$$\tau(a_0, ..., a_n) = \hat{\tau}(a_0 da_1 ... da_n)$$

3. *$b\tau = 0$ and $(1-\lambda)\tau = 0$. That is $\tau \in Z^n_\lambda(A)$.*

Theorems and Examples

Theorem (Connes)

The map $\tau \mapsto \hat{\tau}$ *gives an isomorphism of* $HC^n(A)/\mathrm{Im}B$ *and the quotient space of closed graded traces of degree* n *on* $\Omega^*(A)$ *by those of the form* $d\mu$, *where* μ *is a graded trace of degree* $n+1$ *on* $\Omega^*(A)$.

Theorems and Examples

Theorem (Connes)

Let V be a smooth compact manifold and $A = C^\infty(V)$ the topological space of smooth functions on V. Then

a. *The map $\varphi \mapsto C_\varphi$ is a canonical isomorphism between the continuous Hochschild cohomology $HH(A)$ with the space $\mathcal{D}_k$ of k-dimensional de Rham Currents on V.*

$$\langle C_\varphi, f_0 df_1 \wedge \dots \wedge df_n \rangle = \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon(\sigma) \varphi(f_0, \dots, f_n)$$

b. *Under the isomorphism C, the operator $I \circ B : HH^k(A) \to HH^{k-1}(A)$ is the de Rham boundary, d, for currents.*

Theorems and Examples

From Connes, there exist pairings $\langle K_0(A), HC^e(A)\rangle$ and $\langle K_1(A), HC^o(A)\rangle$ between the first and second K-theory groups of A and the even and odd cyclic cohomological groups of A.

An open question is then, how can we apply these results to manifolds with boundary?

Theorems and Examples

Specifically, if we look at the proof of the previous proposition 1) $\implies$ 3):

$$\tau(a_0, ..., a_n) = \int a_0 da_1 ... da_n$$

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$$= (-1)^{n-1} \int da_n a_0 da_1 ... da_{n-1}$$

Theorems and Examples

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$$\begin{aligned}\tau(a_0, \ldots, a_n) &= \int a_0 da_1 \ldots da_n \\ &= (-1)^{n-1} \int da_n a_0 da_1 \ldots da_{n-1} \\ &= (-1)^{n-1} \int d(a_n a_0) da_1 \ldots da_{n-1} + (-1)^n \int a_n da_0 da_1 \ldots da_{n-1}\end{aligned}$$

Theorems and Examples

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$$\begin{aligned}
\tau(a_0, ..., a_n) &= \int a_0 da_1 ... da_n \\
&= (-1)^{n-1} \int da_n a_0 da_1 ... da_{n-1} \\
&= (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} + (-1)^n \int a_n da_0 da_1 ... da_{n-1} \\
&= (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} + \lambda \tau(a_0, ..., a_n)
\end{aligned}$$

Theorems and Examples

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\tau(a_0, ..., a_n) &= \int a_0 da_1 ... da_n \\
&= (-1)^{n-1} \int da_n a_0 da_1 ... da_{n-1} \\
&= (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} + (-1)^n \int a_n da_0 da_1 ... da_{n-1} \\
&= (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} + \lambda \tau(a_0, ..., a_n)
\end{aligned}$$

Hence

$$(1-\lambda)\tau(a_0, ..., a_n) = (-1)^{n-1} \int d(a_n a_0) da_1 ... da_{n-1} \sim \int_{\partial M} \alpha$$

Theorems and Examples

For example, given a manifold M of dimension n, and $\omega \in \Omega^{n-k}(M)$ a closed $(n-k)$-form, then $\varphi_\omega \in C^\infty(M)^{k+1} \to \mathbb{C}$ given by

$$\varphi_\omega(f_0, \ldots, f_k) = \int_M f_0 df_1 \wedge \ldots \wedge df_k \wedge \omega$$

is cyclic when $\partial M = \emptyset$, i.e. $\varphi(f_0, \ldots, f_k) = (-1)^k \varphi(f_k, f_0, \ldots, f_{k-1})$. This corresponds to the current $C_\omega \in \Omega^k(M) \to \mathbb{C}$ being closed

$$\begin{aligned}
dC_\omega(\eta) &= C_\omega(d\eta) \\
&= \int_M d\eta \wedge \omega \\
&= \int_M d(\eta \wedge \omega) - (-1)^k \int \eta \wedge d\omega \\
&= \int_{\partial M} \eta \wedge \omega - (-1)^k \int \eta \wedge d\omega \\
&= 0
\end{aligned}$$

Theorems and Examples

In other words, a trace that is "closed" and "graded" corresponds to being "cyclic" in cohomology. However, if we have a boundary on our manifold, then by Stokes' theorem we might need to relax the "closed" condition. Instead, we are looking for a complex between the Connes complex $C^\bullet_\lambda(A)$ and the Hochschild complex $C^\bullet(A)$.

For a manifold M with Boundary ∂M, we now have two algebras $A = C^\infty(M)$ and $B = C^\infty(\partial M)$ (or $\mathscr{E}^\infty(\partial M)$) along with a surjection $A \xrightarrow{\sigma} B$ between them, and we are looking for functionals $\varphi \in C^\bullet(A)$ such that $(1 - \lambda)\varphi \in \sigma^* C^\bullet(B)$

Bridge Cohomology

Definitions

Definition (Lesch, Moscovici, Pflaum)

(Originally "Restricted Cyclic Cohomology")
Let A and B be unital $\Bbbk$-algebras, and $\sigma : A \to B$ a surjective unital homomorphism with kernel K. We define the *bridge complex of* σ, $R^n(\sigma)$, to be the complex whose n^{th} module is the set of $\varphi \in C^n(A)$ such that $(1-\lambda)\varphi$ descends do B, meaning

$$(1-\lambda)\varphi = \sigma^*\psi \text{ for some } \psi \in C^n_{\text{bar}}(B).$$

For any $\varphi \in R^n(\sigma)$ we have
$(1-\lambda)b\varphi = b'(1-\lambda)\varphi = b'\sigma^*\psi = \sigma^* b'\psi$ for some $\psi \in C^n(B)$.
Thus b maps $R^n(\sigma)$ to $R^{n+1}(\sigma)$, and $(R^\bullet(\sigma), b)$ is a complex. Its cohomology will be denoted by $HR^\bullet(\sigma)$ and called the *bridge cohomology*.

Proposition

We have a non-direct sum $R^\bullet(\sigma) = C_\lambda^\bullet(A) + \sigma^* C^\bullet(B)$.

Proof.

Given any $\varphi \in R^n(\sigma)$, let $\psi \in C^n_{\mathrm{bar}}(B)$ be such that $(1-\lambda)\varphi = \sigma^*\psi$. Notice that $Q\sigma^*\psi = Q(1-\lambda)\varphi = 0$ we have $\sigma^* Q\psi = 0$, and hence $Q\psi = 0$ since σ^* is injective. Thus, there exists $\psi' \in C^n(B)$ such that $(1-\lambda)\psi' = \psi$. Now we write $\varphi = (\varphi - \sigma^*\psi') + \sigma^*\psi'$. □

Bridge Cohomology

Categorical Constructions

We define the category $\mathcal{S}_{\mathbb{1},\Bbbk}$ to have objects as surjective unital $\Bbbk$-algebra homomorphisms. Given two objects $\sigma, \tau \in \mathrm{Obj}(\mathcal{S}_{\mathbb{1},\Bbbk})$, a morphism from σ to τ is a pair of unital algebra homomorphisms (f_1, f_2) such that $\tau f_1 = f_2 \sigma$ i.e. we have a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ X & \xrightarrow{\tau} & Y \end{array}$$

Note that this is a monoidal category where the unit is $\mathrm{id}_{\Bbbk} : \Bbbk \to \Bbbk$, which for any $\sigma : A \to B$, we get the canonical inclusion $\mathrm{id}_{\Bbbk} \xrightarrow{(\iota_A, \iota_B)} \sigma$.

We can now define the bridge complex as a (contravariant) functor $R^\bullet(\cdot) : \mathcal{S}_{\mathbb{1},\Bbbk} \to \mathcal{C}$, from the category of surjective unital algebra homomorphisms to the category of chain complexes.

Of special note:

$$\begin{aligned} R^\bullet(\mathrm{id}_A) &= \{\varphi \in C^\bullet(A) \mid (1-\lambda)\varphi = \mathrm{id}_A^*\psi \text{ for some } \psi \in C^\bullet(A)\} \\ &= C^\bullet(A) \end{aligned}$$

And for the zero map we have the short exact sequence $A \to A \xrightarrow{0_A} 0$, and bridge cohomology

$$R^\bullet(0_A) = \{\varphi \in C^\bullet(A) \mid (1-\lambda)\varphi = 0\} = C^\bullet_\lambda(A).$$

So for any map σ with domain A, we have $C^\bullet_\lambda(A) \subset R^\bullet(\sigma) \subset C^\bullet(A)$, and the degree to how close $R^\bullet(\sigma)$ sits between either two is measured in some sense by the size of the kernel of σ.

Definition

Given any $\Bbbk$-algebras A and B (not necessarily unital) and a surjective algebra homomorphism $\sigma : A \to B$, let $\sigma_+ : A_+ \to B_+$ be the extension of σ to the augmented algebras A_+ and B_+. We define the n^{th} bridge cohomology module of σ as:

$$HR^n(\sigma) := \ker \left(HR^n(\sigma_+) \xrightarrow{\iota^*} HR^n(\mathrm{id}_\Bbbk) \right).$$

Definition

For a surjective map of unital algebras $A \xrightarrow{\sigma} B$, the bridge complex $R(\sigma)$ can be defined as the pullback in the following diagram:

$$\begin{array}{ccc} R^\bullet(\sigma) & \longrightarrow & C^\bullet(A) \\ \downarrow & \ulcorner & \downarrow {\scriptstyle 1-\lambda} \\ C^\bullet_{\mathrm{bar}}(B) & \xrightarrow{\sigma^*} & C^\bullet_{\mathrm{bar}}(A) \end{array}$$

In the category of cochain complexes, this means that

$$R^n(\sigma) = \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C^n(A) \times C^n_{\mathrm{bar}}(B) \,\middle|\, (1-\lambda)\varphi = \sigma^*\psi \right\},$$

with differential $\begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}$, though since the map σ^* is injective, including the bottom entry ψ isn't quite necessary, so we will often only write elements as $\varphi \in R^n(\sigma)$.

Definition

We may define the *reduced bridge complex* $R(\sigma)_{\mathrm{red}}$ as either of the following kernels:

$$0 \to R(\sigma)_{\mathrm{red}} \to R(\sigma) \to D(\sigma)_{\mathrm{red}} \to 0$$

or

$$0 \to R(\sigma)_{\mathrm{red}} \to \overline{R}(\sigma) \to \mathring{\mathbb{j}}_A[0] \to 0$$

It's cohomology will be denoted $\overline{HR}^n(\sigma) = H^n(R(\sigma)_{\mathrm{red}})$. We a long exact sequence:

$$0 \to \overline{HR}^0(\sigma) \to HR^0(\sigma) \to \mathring{\mathbb{j}}_A \to \overline{HR}^1(\sigma) \to HR^1(\sigma) \to 0 \to \ldots$$
$$\to \overline{HR}^{2n}(\sigma) \to HR^{2n}(\sigma) \to \mathring{\mathbb{j}}_A/\mathring{\mathbb{j}}_B \to \overline{HR}^{2n+1}(\sigma) \to HR^{2n+1}(\sigma) \to 0 \to$$

Bridge Cohomology

Normalized and Reduced Complexes

Proposition

$R(\sigma)_{\mathrm{red}}$ is the pullback of the corresponding normalized complexes:

$$
\begin{array}{ccc}
R^\bullet(\sigma)_{\mathrm{red}} & \longrightarrow & C^\bullet(A)_{\mathrm{red}} \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle 1-\lambda} \\
C^\bullet_{\mathrm{bar}}(B)_{\mathrm{red}} & \xrightarrow{\sigma^*} & C^\bullet_{\mathrm{bar}}(A)_{\mathrm{red}}
\end{array}
$$

Proof.

$$
\begin{array}{ccc}
C^\bullet(\sigma)_{\mathrm{red}} & \longrightarrow & C^\bullet(A)_{\mathrm{red}} \\
\big\downarrow & & \big\downarrow {\scriptstyle 1-\lambda} \\
C^\bullet_{\mathrm{bar}}(B)_{\mathrm{red}} & \xrightarrow{\sigma^*} & C^\bullet_{\mathrm{bar}}(A)_{\mathrm{red}}
\end{array}
\rightsquigarrow
\begin{array}{ccc}
R^\bullet(\sigma) & \longrightarrow & C^\bullet(A) \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle 1-\lambda} \\
C^\bullet_{\mathrm{bar}}(B) & \xrightarrow{\sigma^*} & C^\bullet_{\mathrm{bar}}(A)
\end{array}
\rightsquigarrow
\begin{array}{ccc}
D^\bullet(\sigma) & \longrightarrow & D^\bullet(A) \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle 1-\lambda} \\
D^\bullet_{\mathrm{bar}}(B) & \xrightarrow{\sigma^*} & D^\bullet_{\mathrm{bar}}(A)_{\mathrm{red}}
\end{array}
$$

□

Bridge Cohomology

Non-unital Algebras

Theorem

For a non-unital surjection $\sigma : A \to B$*,* $HR^n(\sigma) = H^n(R(\sigma_+)_{\mathrm{red}})$*.*

Proof.

It should be clear that $\mathring{\mathbb{j}}_{A_+} = \mathring{\mathbb{j}}_{B_+} = \Bbbk$, and $R(\mathrm{id}_\Bbbk) = C(\Bbbk)$. Hence $HR^n(\mathrm{id}_\Bbbk) = HH^n(\Bbbk) = \begin{cases} 0 & n > 0 \\ \Bbbk & n = 0 \end{cases}$. The previous long exact sequence then becomes

$$0 \to \overline{HR}^0(\sigma_+) \to HR^0(\sigma_+) \to \Bbbk \to \overline{HR}^1(\sigma_+) \to HR^1(\sigma_+) \to 0 \to \dots$$

Note that in degree 0, $R^0(\sigma_+) = (A_+)^*$ and that the map $HR^0(\sigma_+) \to \Bbbk$ is induced by the evaluation map $ev_1 : (A_+)^* \to \Bbbk$ which is surjective, since the projection $\pi : A_+ \to \Bbbk$ is such that $b\pi = 0$. Thus the result follows. □

Definition

The bicomplex $RB(\sigma)^{\{2\}}$ associated to $R(\sigma_+)_{\mathrm{red}}$ is given by the pullback

$$\begin{array}{ccc} RB(\sigma)^{\{2\}} & \longrightarrow & CC(A)^{\{2\}} \\ \downarrow & \ulcorner & \downarrow {\scriptstyle 1-\lambda \oplus 1} \\ CB(B)^{\{2\}} & \xrightarrow{\sigma^* \oplus \sigma^*} & CB(A)^{\{2\}} \end{array}$$

Corollary

$H^n(\mathrm{Tot}\, RB(\sigma)^{\{2\}}) = HR^n(\sigma).$

Bridge Cohomology

Non-unital Algebras

Explicitly, we get the following diagram for $RB(\sigma)^{\{2\}}$

$$
\begin{array}{ccc}
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} \\
R^2(\sigma) & \xrightarrow{(1-\lambda)^{\nabla}} & C^2_{\mathrm{bar}}(B) \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} \\
R^1(\sigma) & \xrightarrow{(1-\lambda)^{\nabla}} & C^1_{\mathrm{bar}}(B) \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} \\
R^0(\sigma) & \xrightarrow{(1-\lambda)^{\nabla}} & C^0_{\mathrm{bar}}(B)
\end{array}
$$

where $(1-\lambda)^{\nabla} = (\sigma^*)^{-1}(1-\lambda)$.

Let $\sigma : A \to A/I$ be a surjective unital algebra homomorphism, where $I \subset A$ is an ideal. Then the relative Hochschild complex $C(A, I)$ is defined as the cokernel

$$0 \to C(A/I) \to C(A) \to C(A, I) \to 0.$$

It's cohomology will be denoted $HH(A, I)$ called the relative Hochschild homology, and we of course have a long exact sequence relating the three

$$\ldots \to HH^n(A, I) \to HH^n(A) \to HH^n(A/I) \to HH^{n+1}(A, I) \to \ldots$$

Bridge Cohomology

Cyclic bicomplexes

Definition

Let A be a unital algebra and $0 \to I \to A \xrightarrow{\sigma} A/I \to 0$ be a short exact sequence of algebras. Define the *bridge bicomplex of* σ, $RR(\sigma)$, as the bicomplex with the following columns

$$C(A) \xrightarrow{q(1-\lambda)} C_{\mathrm{bar}}(A,I) \xrightarrow{Q} C(A,I) \xrightarrow{1-\lambda} C_{\mathrm{bar}}(A,I) \xrightarrow{Q} \ldots$$

We will denote the cohomology of the total complex by $HR^n(\sigma) := H^n(\mathrm{Tot}\, RR(\sigma))$ and call it the *bridge cohomology of* σ.

Bridge Cohomology

Cyclic bicomplexes

$$
\begin{array}{cccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
C^2(A) & \longrightarrow & C^2_{\mathrm{bar}}(A, I) & \longrightarrow & C^2(A, I) & \longrightarrow & C^2_{\mathrm{bar}}(A, I) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
C^1(A) & \longrightarrow & C^1_{\mathrm{bar}}(A, I) & \longrightarrow & C^1(A, I) & \longrightarrow & C^1_{\mathrm{bar}}(A, I) & \longrightarrow \\
b \uparrow & & -b' \uparrow & & \uparrow & & \uparrow & \\
C^0(A) & \xrightarrow[q(1-\lambda)]{} & C^0_{\mathrm{bar}}(A, I) & \xrightarrow[Q]{} & C^0(A, I) & \xrightarrow[1-\lambda]{} & C^0_{\mathrm{bar}}(A, I) & \longrightarrow
\end{array}
$$

Proposition

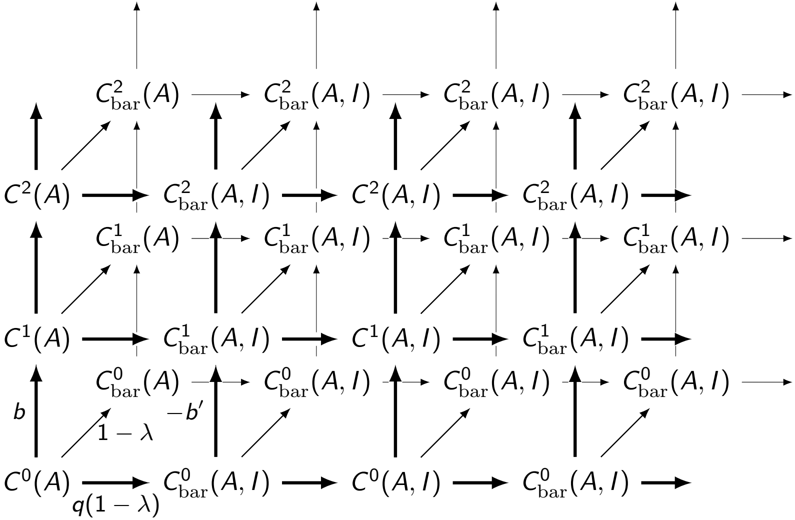
When $\Bbbk$ contains $\mathbb{Q}$, the total complex of $RR(\sigma)$ is quasi-isomorphic to the bridge complex, $\mathrm{Tot}\, RR(\sigma) \overset{q}{\cong} R(\sigma)$.

Proposition

For a non-unital surjection $A \xrightarrow{\sigma} B$, $\overline{HR}(\sigma_+) = HR(\sigma)$.

Lemma

For an augmented morphism $\sigma_+ : A_+ \to A_+/I$,
$HR^n(\sigma) = H^n(\mathrm{Tot}\, RRB(\sigma))$, *where* $RRB(\sigma)$ *is the following tricomplex*



Definition

For any algebra surjection $\sigma : A \to B$, the *bar bicomplex of* σ, $RR_{\mathrm{bar}}(\sigma)$, is the back sheet of $RRB(\sigma)$. That is, $RR_{\mathrm{bar}}(\sigma)$ is the bicomplex with columns

$$RR_{\mathrm{bar}}(\sigma) := C_{\mathrm{bar}}(A) \xrightarrow{q} C_{\mathrm{bar}}(A, I) \xrightarrow{0} C_{\mathrm{bar}}(A, I) \xrightarrow{1} C_{\mathrm{bar}}(A, I) \xrightarrow{0} \ldots$$

The *bar cohomology of* σ, $HB^n(\sigma)$, is given as the total cohomology of this complex. $HB^n(\sigma) := H^n(\mathrm{Tot}\, RR_{\mathrm{bar}}(\sigma))$.

Definition

An algebra surjection $\sigma : A \to B$ is said to be *coH-unital* when $HB^n(\sigma) = 0$ for all n.

Given an exact sequence in $\mathcal{S}_{\Bbbk}$,

$$0 \to \sigma \xrightarrow{(f_1,f_2)} \tau \xrightarrow{(g_1,g_2)} \tau/\sigma \to 0$$

we can define the relative bridge cocomplex, $R^\bullet(\tau,\sigma)$, as the cokernel

$$\begin{array}{ccccccccc} & & & & & & R^\bullet(\sigma) & & \\ & & & & & \nearrow & \uparrow & & \\ 0 & \longrightarrow & R^\bullet(\tau/\sigma) & \longrightarrow & R^\bullet(\tau) & \longrightarrow & R^\bullet(\tau,\sigma) & \longrightarrow & 0 \end{array}$$

Theorem (Excision (conjectured))

Given an exact sequence in $\mathcal{S}_{\Bbbk}$,

$$0 \to \sigma \xrightarrow{(f_1, f_2)} \tau \xrightarrow{(g_1, g_2)} \tau/\sigma \to 0$$

with τ and τ/σ unital, then the map $R^\bullet(\tau, \sigma) \to R^\bullet(\sigma)$ is a quasi-isomorphism if and only if σ is coH-unital.

Research Goal

Generalize the Gysin-Connes sequence to bridge cohomology

$$\ldots \xrightarrow{I} HH^{n-1}(A) \xrightarrow{B} HC^{n-2}(A) \xrightarrow{S} HC^{n}(A) \xrightarrow{I} HH^{n}(A) \to \ldots$$

Research Goal

Correlate bridge cohomology and de Rham Homology on manifolds with boundary: (L.,M.,P.) for M compact and $C^\infty(M) \xrightarrow{\sigma} \mathscr{E}^\infty(\partial M)$,

$$HR(\sigma) \cong B^{-1}(\mathscr{D}'_{k-1}(M;\partial M)) \oplus H^{dR}_{k-2}(M;\partial M) \oplus H^{dR}_{k-4}(M;\partial M) \oplus \ldots$$

Extend the pairings $\langle K_0(A), HC^e(A)\rangle$ *and* $\langle K_1(A), HC^o(A)\rangle$ *from Connes, to manifolds with boundaries.*

Research Goal (Exterior Differential Systems)

Given a system of PDE's, $F^k(x, y, \frac{\partial^{|\alpha|}}{\partial x^\alpha}) = 0$*, we can reformulate the problem of looking for solutions to this system in terms of looking for integral submanifolds* $N \xrightarrow{i} M$*, such that* $i^*\mathcal{I} = 0$*, where* M *is some suitably chosen jet space, and* $\mathcal{I}$ *is a differential ideal*

$$0 \to \mathcal{I} \to \Omega(M) \to \Omega(M)/\mathcal{I} \to 0$$

that corresponds with the original system of PDE's.

Using the techniques developed, we now have a cohomology theory to apply to this situation. The question is, what type of information (if any) in terms of integrability can determined by it's cohomology groups?

Thank You!